7.4 - Operational Properties II

Derivatives of Transforms

The formulas we saw before $\mathscr{L}\{y'\} = sY(s) - y(0)$ and $\mathscr{L}\{y''\} = s^2Y(s) - sy(0) - y'(0)$ are transforms of derivatives. Now we will consider *derivatives of transforms*.

Consider $F(s) = \int_0^\infty e^{-st} f(t)\,dt$

Differentiating WRT s yields

$$\frac{d}{ds}[F(s)] = \frac{d}{ds}\left[\int_0^\infty e^{-st} f(s)\,dt\right]$$

$$= \int_0^\infty \frac{\partial}{\partial s}(e^{-st}) f(t)\,dt = -\int_0^\infty e^{-st}\,\boxed{t\,f(t)}\,dt$$

$$\frac{d}{ds}[F(s)] = -\mathscr{L}\{t f(t)\}$$

$$\mathscr{L}\{tf(t)\} = -\frac{d}{ds}[F(s)]$$

Example: Evaluate the given Laplace transform.

$\mathscr{L}\{t \sin t\}$

$$= -\frac{d}{ds}\left[\frac{1}{s^2+1}\right]$$

$$= -\frac{d}{ds}\left[(s^2+1)^{-1}\right]$$

$$= \frac{2s}{(s^2+1)^2}$$

$\mathscr{L}\{te^{-3t}\sin t\}$

$-\frac{d}{ds}$ shift what we're transforming

$$= -\frac{d}{ds}\left[\frac{1}{(s+3)^2+1}\right]$$

$$= -\frac{d}{ds}\left[\left[(s+3)^2+1\right]^{-1}\right]$$

$$= \frac{2(s+3)}{\left[(s+3)^2+1\right]^2}$$

$$\mathcal{L}\{t^2 f(t)\} = \mathcal{L}\{\underline{t} \cdot t f(t)\} = -\frac{d}{ds}\left[\mathcal{L}\{\underline{t} f(t)\}\right]$$

$$= -\frac{d}{ds}\left[-\frac{d}{ds}\left[F(s)\right]\right]$$

$$= \frac{d^2}{ds^2}\left[F(s)\right]$$

In general, we have

Theorem 7.4.1: Derivatives of Transforms

If $F(s) = \mathscr{L}\{f(t)\}$ and $n = 1,\ 2,\ 3, \ldots$, then $\mathscr{L}\{t^n f(t)\} = (-1)^n \dfrac{d^n}{ds^n} F(s)$.

Example: Use Theorem 7.4.1 to evaluate the given Laplace transform.

$\mathscr{L}\{t^2 \cos t\}$

$\frac{d^2}{ds^2}$

$$= \frac{d^2}{ds^2}\left(\frac{s}{s^2+1}\right) = \frac{d}{ds}\left(\frac{s^2+1-2s^2}{(s^2+1)^2}\right)$$

$$= \frac{d}{ds}\left(\frac{1-s^2}{(s^2+1)^2}\right)$$

$$= \frac{-2s(s^2+1)^{\cancel{2}} + (s^2-1)\cdot 2\cancel{(s^2+1)}\cdot 2s}{(s^2+1)^{\cancel{4}\,3}}$$

$$F(s) = \frac{2s^3-6s}{(s^2+1)^3}$$

Consider $\mathcal{L}\{t e^{at}\}$

Example: Use the Laplace transform to solve the given initial-value problem. Use the table of Laplace transforms in Appendix C as needed.

$y' - y = te^t \sin t, \qquad y(0) = 0$

$$sY(s) - Y(s) = -\frac{d}{ds}\left(\frac{1}{(s-1)^2+1}\right)$$

Work goes here

$$Y(s) = \frac{2\cancel{(s-1)}}{[(s-1)^2+1]^2\cancel{(s-1)}}$$

#25 $k=1$

$$\boxed{y(t) = e^t(\sin t - t\cos t)}$$

$y'' + y = f(t), \qquad y(0) = 1, \quad y'(0) = 0$, where $f(t) = \begin{cases} 1, & 0 \le t < \pi/2 \\ \sin t, & t \ge \pi/2 \end{cases}$

$$f(t) = \begin{cases} 1, & 0 \le t < \pi/2 \\ 0, & t \ge \pi/2 \end{cases} + \begin{cases} 0, & 0 \le t < \pi/2 \\ \sin t, & t \ge \pi/2 \end{cases}$$

$$= 1 - \mathcal{U}(t - \pi/2) + \sin t\, \mathcal{U}(t - \pi/2)$$

Compensation

$$y'' + y = 1 - \mathcal{U}(t-\pi/2) + \sin t\, \mathcal{U}(t - \pi/2)$$

$$s^2 Y(s) - s + Y(s) = \frac{1}{s} - \frac{e^{-\pi/2 s}}{s} + e^{-\pi/2 s}\mathcal{L}\{\sin(t+\pi/2)\}$$

$$\sin(t+\pi/2) = \sin t \cancel{\cos}^{\,0} \pi/2 + \cos t \sin \pi/2 = \cos t$$

$$s^2Y(s) - s + Y(s) = \frac{1}{s} - \frac{e^{-\pi/2 s}}{s} + \frac{se^{-\pi/2 s}}{s^2+1}$$

$$(s^2+1)Y(s) = \underbrace{\frac{1}{s} + s}_{\frac{s^2+1}{s}} + e^{-\pi/2 s}\left(\frac{s}{s^2+1} - \frac{1}{s}\right)$$

$$Y(s) = \frac{1}{s} + e^{-\pi/2\, s}\left(\frac{s}{(s^2+1)^2} - \frac{1}{(s^2+1)s}\right)$$

$$\frac{1}{s(s^2+1)} = \frac{A}{s} + \frac{Bs+C}{s^2+1} = \frac{s}{s^2+1} - \frac{1}{s}$$

Note this has the form of a derivative, so we can use $\mathcal{L}^{-1}\left\{-\frac{d}{ds}[F(s)]\right\} = t f(t)$

$$Y(s) = \mathcal{L}^{-1}\left\{\frac{1}{s}\right\} + \mathcal{L}^{-1}\left\{e^{-\pi/2\, s}\frac{s}{(s^2+1)^2}\right\} + \mathcal{L}^{-1}\left\{e^{-\pi/2\, s}\left(\frac{s}{s^2+1} - \frac{1}{s}\right)\right\}$$

$$y(t) = 1 + \frac{1}{2}\mathcal{U}(t-\pi/2)(t-\pi/2)\sin(t-\pi/2) + \mathcal{U}(t-\pi/2)[\cos(t-\pi/2) - 1]$$

$$\boxed{y(t) = 1 + \left[-\sin t - 1 - \frac{1}{2}(t-\pi/2)\cos t\right]\mathcal{U}(t-\pi/2)}$$

Convolution

Definition: The **convolution** of f and g is $f * g = \int_0^t f(\tau)g(t-\tau)\,d\tau$

tau

the result is a function of t.

It can be shown that $f*g = g*f$

Example: Find $\cos 2t * e^t$. $= \int_0^t \cos 2\tau \, e^{t-\tau} \, d\tau$

$$= e^t \int_0^t \cos 2\tau \, e^{-\tau} \, d\tau$$

$u = \cos 2\tau$ $\quad dv = e^{-\tau} d\tau$

$du = -2\sin 2\tau \, d\tau$ $\quad v = -e^{-\tau}$

$$= e^t \left(-e^{-\tau} \cos 2\tau \Big|_0^t + \int_0^t 2 \sin 2\tau \, e^{-\tau} \, d\tau \right)$$

$-e^{-t} \cos 2t + 1$

After a moderate amount of work,

$$\boxed{\cos 2t * e^t = \frac{1}{5} e^t - \frac{1}{5} \cos 2t + \frac{2}{5} \sin 2t}$$

Example: Find $\mathscr{L}\{\cos 2t * e^t\}$. $= \dfrac{1/5}{s-1} - \dfrac{1/5\, s}{s^2+4} + \dfrac{4/5}{s^2+4}$

$$= \frac{1}{5}\,\frac{s^2+4-s^2+s+4s-4}{(s-1)(s^2+4)}$$

$$= \frac{s}{(s-1)(s^2+4)} = \frac{s}{s^2+4}\cdot\frac{1}{s-1}$$

$$\mathscr{L}\{\cos 2t * e^t\} = \mathscr{L}\{\cos 2t\}\,\mathscr{L}\{e^t\}$$

In general,

Theorem: Convolution Theorem

If $f(t)$ and $g(t)$ are piecewise continuous on $[0,\infty)$ and of exponential order, then $\mathscr{L}\{f * g\} = \mathscr{L}\{f(t)\}\mathscr{L}\{g(t)\} = F(s)G(s)$.

Example: Find the Laplace transform of $f * g$ using the Convolution Theorem. Do not evaluate the convolution integral before transforming.

$$\mathscr{L}\left\{t^3 * t\sin 3t\right\} = \mathcal{L}\{t^3\}\,\mathcal{L}\{t\sin 3t\}$$

$$= \frac{6}{s^4}\left(-\frac{d}{ds}\left(\frac{3}{s^2+9}\right)\right) = \frac{36}{s^3(s^2+9)^2}$$

$\sin t * \cos t$

$$\mathscr{L}\left\{\int_0^t \sin\tau\cos(t-\tau)\,d\tau\right\}$$

$$= \mathcal{L}\{\sin t\}\,\mathcal{L}\{\cos t\}$$

$$= \frac{s}{(s^2+1)^2}$$

$e^{-t}\cos t * 1$

$$\mathscr{L}\left\{\int_0^t \underbrace{e^{-\tau}\cos\tau}_{f(\tau)}\cdot 1\,d\tau\right\}$$

$g(t-\tau)=1$

$$= \mathcal{L}\{e^{-t}\cos t\}\,\mathcal{L}\{1\}$$

$$= \frac{s+1}{\left((s+1)^2+1\right)s}$$

The transform of an integral

When $g(t) = 1$, $\mathscr{L}\{g(t)\} = G(s) = 1/s$, the convolution theorem gives

$\mathscr{L}\left\{\int_0^t f(\tau)\,d\tau\right\} = \dfrac{F(s)}{s}$, which in inverse form is $\int_0^t f(\tau)\,d\tau = \mathscr{L}^{-1}\left\{\dfrac{F(s)}{s}\right\}$.

Example: Evaluate the given inverse transform.

$$\mathscr{L}^{-1}\left\{\frac{1}{s(s-a)^2}\right\}$$

Example: Use the Laplace transform to solve the given integral equation.

$$f(t) + 2\int_0^t f(\tau)\cos(t-\tau)\,d\tau = 4e^{-t} + \sin t$$

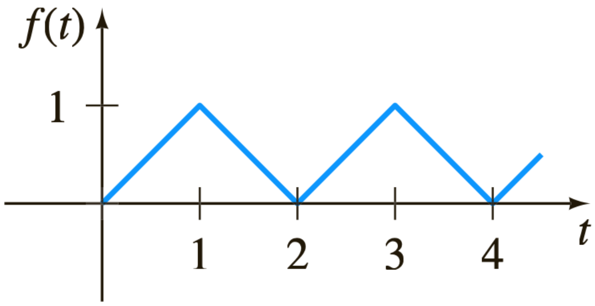
Example: Use the Laplace transform to solve the given integrodifferential equation.

$$\frac{dy}{dt} + 6y(t) + 9\int y(\tau)\,d\tau = 1, \qquad y(0) = 0$$

Theorem: Transform of a Periodic Function

If $f(t)$ is piecewise continuous on $[0,\ \infty)$, of exponential order, and periodic with period T, then $\mathscr{L}\{f(t)\} = \dfrac{1}{1 - e^{-sT}} \int_0^T e^{-st} f(t)\,dt$.

Example: Find the Laplace transform of the given periodic function.



triangular wave